

TWISTORS IN CONFORMALLY FLAT EINSTEIN FOUR-MANIFOLDS

Giampiero Esposito^{1,2} and Giuseppe Pollifrone³

¹*Istituto Nazionale di Fisica Nucleare, Sezione di Napoli*

Mostra d'Oltremare Padiglione 20, 80125 Napoli, Italy;

²*Dipartimento di Scienze Fisiche*

Mostra d'Oltremare Padiglione 19, 80125 Napoli, Italy;

³*Dipartimento di Fisica, Università di Roma "La Sapienza"*

and INFN, Sezione di Roma, Piazzale Aldo Moro 2, 00185 Roma, Italy.

Abstract. This paper studies the two-component spinor form of massive spin- $\frac{3}{2}$ potentials in conformally flat Einstein four-manifolds. Following earlier work in the literature, a non-vanishing cosmological constant makes it necessary to introduce a supercovariant derivative operator. The analysis of supergauge transformations of primary and secondary potentials for spin $\frac{3}{2}$ shows that the gauge freedom for massive spin- $\frac{3}{2}$ potentials is generated by solutions of the supertwistor equations. The supercovariant form of a partial connection on a non-linear bundle is then obtained, and the basic equation of massive secondary potentials is shown to be the integrability condition on super β -surfaces of a differential operator on a vector bundle of rank three. Moreover, in the presence of boundaries, a simple algebraic relation among some spinor fields is found to ensure the gauge invariance of locally supersymmetric boundary conditions relevant for quantum cosmology and supergravity.

PACS numbers: 0420, 0465, 9880

1. Introduction

The local theory of $\text{spin-}\frac{3}{2}$ potentials in real Riemannian 4-geometries is receiving careful consideration in the current literature. There are at least two motivations for this analysis. In Minkowski space-time, twistors arise naturally as charges for massless $\text{spin-}\frac{3}{2}$ fields [1-5]. In Ricci-flat 4-manifolds, such fields are well defined (Ricci-flatness being a necessary and sufficient consistency condition), and a suitable generalization of the concept of twistors would make it possible to reconstruct solutions of the vacuum Einstein equations out of the resulting twistor space. In extended supergravity theories, however, it is necessary to make sense of massive $\text{spin-}\frac{3}{2}$ fields in Riemannian backgrounds. For this purpose, a careful spinorial analysis of the problem is in order.

We have thus focused on massive $\text{spin-}\frac{3}{2}$ potentials in 4-manifolds with non-vanishing cosmological constant, considering the supercovariant derivative compatible with a non-vanishing scalar curvature. This is the content of section 2. Section 3 studies the gauge freedom of the second kind, which is generated by a particular type of twistors, i.e. the Euclidean Killing spinors. Section 4 studies the preservation of $\text{spin-}\frac{3}{2}$ field equations under the supergauge transformations of primary potentials. Section 5 studies secondary potentials for $\text{spin-}\frac{3}{2}$ in the massive case. In section 6 a partial superconnection acting on a bundle with non-linear fibres is introduced. Section 7 studies the action of a superconnection on a vector bundle of rank three, and the corresponding integrability condition on super β -surfaces is derived. Section 8 studies the case of backgrounds with boundaries and shows the conditions under which locally supersymmetric boundary conditions are

gauge-invariant. Results and open problems are described in section 9. Relevant details are given in the appendix.

2. The superconnection

In the massless case, the two-spinor form of the Rarita-Schwinger equations is the one given in the appendix, where $\nabla_{AA'}$ is the spinor covariant derivative corresponding to the connection ∇ of the background. In the massive case, however, the appropriate connection, hereafter denoted by S , has an additional term which couples to the cosmological constant $\lambda = 6\Lambda$ [9,10]. In the language of γ -matrices, the new covariant derivative S_μ to be inserted *in the field equations* takes the form [9,10]

$$S_\mu \equiv \nabla_\mu + f(\Lambda)\gamma_\mu \quad (2.1)$$

where $f(\Lambda)$ vanishes at $\Lambda = 0$, and γ_μ are the curved-space γ -matrices. Since, following [1-8], we are interested in the two-spinor formulation of the problem, we have to bear in mind the action of γ -matrices on any spinor $\varphi \equiv (\beta^C, \tilde{\beta}_{C'})$. Note that unprimed and primed spin-spaces are no longer (anti)-isomorphic in the case of positive-definite 4-metrics, since there is no complex conjugation which turns primed spinors into unprimed spinors (or the other way around) [5,11]. Hence β^C and $\tilde{\beta}_{C'}$ are totally unrelated. With this understanding, we write the supergauge transformations for massive spin- $\frac{3}{2}$ potentials in the form (cf [1-5])

$$\hat{\gamma}^A_{B'C'} \equiv \gamma^A_{B'C'} + S^A_{B'} \lambda_{C'} \quad (2.2)$$

$$\widehat{\Gamma}^{A'}_{BC} \equiv \Gamma^{A'}_{BC} + S^{A'}_B \nu_C \quad (2.3)$$

where the action of $S_{AA'}$ on the gauge fields $(\nu^B, \lambda_{B'})$ is defined by (cf (2.1))

$$S_{AA'} \nu_B \equiv \nabla_{AA'} \nu_B + f_1(\Lambda) \epsilon_{AB} \lambda_{A'} \quad (2.4)$$

$$S_{AA'} \lambda_{B'} \equiv \nabla_{AA'} \lambda_{B'} + f_2(\Lambda) \epsilon_{A'B'} \nu_A. \quad (2.5)$$

With our notation, $R = 24\Lambda$ is the scalar curvature, f_1 and f_2 are two functions which vanish at $\Lambda = 0$, whose form will be determined later by a geometric analysis. The action of $S_{AA'}$ on a many-index spinor $T^{A...L}_{B'...F'}$ can be obtained by expanding such a T as a sum of products of spin-vectors, i.e. [12]

$$T^{A...L}_{B'...F'} = \sum_i \alpha^A_{(i)} \dots \beta^L_{(i)} \gamma^{(i)}_{B'} \dots \delta^{(i)}_{F'} \quad (2.6)$$

and then applying the Leibniz rule and the definitions (2.4)-(2.5), where $\alpha^A_{(i)}$ has an independent partner $\widetilde{\alpha}^{A'}_{(i)}$, ... , $\gamma^{(i)}_{B'}$ has an independent partner $\widetilde{\gamma}^{(i)}_B$, ... , and so on. A further, non-trivial requirement is that $S_{AA'}$ should annihilate the curved ϵ -spinors [12], in much the same way as $\nabla_{AA'}$ annihilates such spinors. In our analysis we always assume that

$$S_{AA'} \epsilon_{BC} = 0 \quad (2.7)$$

$$S_{AA'} \epsilon_{B'C'} = 0. \quad (2.8)$$

In the light of the definitions and assumptions presented so far, one can make sense of the Rarita-Schwinger equations with non-vanishing cosmological constant $\lambda = 6\Lambda$, i.e. (cf appendix)

$$\epsilon^{B'C'} S_{A(A'} \gamma^A_{B')C'} = \Lambda \widetilde{F}_{A'} \quad (2.9)$$

$$S^{B'(B} \gamma^{A)}_{B'C'} = 0 \quad (2.10)$$

$$\epsilon^{BC} S_{A'(A} \Gamma^{A'}_{B)C} = \Lambda F_A \quad (2.11)$$

$$S^{B(B'} \Gamma^{A')}_{BC} = 0. \quad (2.12)$$

With our notation, F_A and $\tilde{F}_{A'}$ are spinor fields proportional to the traces of secondary potentials for spin $\frac{3}{2}$. These will be studied in section 5.

3. Gauge freedom of the second kind

The gauge freedom of the second kind is the one which does not affect the potentials after a gauge transformation. This requirement corresponds to the case analyzed in [13], where it is pointed out that whilst the Lagrangian of $N = 1$ supergravity is invariant under gauge transformations with arbitrary spinor fields $(\nu^A, \lambda_{A'})$, the actual *solutions* are only invariant if the supercovariant derivatives (2.4)-(2.5) vanish.

On setting to zero $S_{AA'} \nu_B$ and $S_{AA'} \lambda_{B'}$, one gets a coupled set of equations which are the Euclidean version of the Killing-spinor equation [13], i.e.

$$\nabla^{A'}_B \nu_C = -f_1(\Lambda) \lambda^{A'} \epsilon_{BC} \quad (3.1)$$

$$\nabla^A_{B'} \lambda_{C'} = -f_2(\Lambda) \nu^A \epsilon_{B'C'}. \quad (3.2)$$

What is peculiar of equations (3.1)-(3.2) is that their right-hand sides involve spinor fields which are, themselves, solutions of the twistor equation. Hence one deals with a special

type of twistors, which do not exist in a generic curved manifold (cf [13]). Equation (3.1) can be solved for $\lambda^{A'}$ as

$$\lambda_{C'} = \frac{1}{2f_1(\Lambda)} \nabla_{C'}{}^B \nu_B. \quad (3.3)$$

The insertion of (3.3) into (3.2) and the use of spinor Ricci identities [5,12] yields the second-order equation

$$\square \nu_A + (6\Lambda + 8f_1 f_2) \nu_A = 0. \quad (3.4)$$

On the other hand, (3.1) implies the twistor equation

$$\nabla^{A'}{}_{(B} \nu_{C)} = 0. \quad (3.5)$$

Covariant differentiation of (3.5), jointly with spinor Ricci identities, leads to [8]

$$\square \nu_A - 2\Lambda \nu_A = 0. \quad (3.6)$$

By comparison of (3.4) and (3.6) one finds the condition $f_1 f_2 = -\Lambda$. The integrability condition of (3.5) is given by [11]

$$\psi_{ABCD} \nu^D = 0. \quad (3.7)$$

This means that our manifold is conformally left-flat, unless ν^D is a four-fold principal spinor of the anti-self-dual Weyl spinor. The latter possibility is here ruled out, to avoid having gauge fields related explicitly to the curvature of the background.

The condition $f_1 f_2 = -\Lambda$ is also obtained by comparison of first-order equations, since for example

$$\nabla^{AA'} \nu_A = 2f_1 \lambda^{A'} = -2 \frac{\Lambda}{f_2} \lambda^{A'}. \quad (3.8)$$

The first equality in (3.8) results from (3.1), whilst the second one is obtained since the twistor equations also imply that (see (3.2))

$$\nabla^{AA'} \left(-f_2 \nu_A \right) = 2\Lambda \lambda^{A'}. \quad (3.9)$$

Entirely analogous results are obtained on considering the twistor equation resulting from (3.2), i.e.

$$\nabla^A_{(B'} \lambda_{C')} = 0. \quad (3.10)$$

The integrability condition of (3.10) is

$$\tilde{\psi}_{A'B'C'D'} \lambda^{D'} = 0. \quad (3.11)$$

Since our gauge fields are not assumed to be four-fold principal spinors of the Weyl spinor (cf [14]), equations (3.7) and (3.11) imply that our background geometry is conformally flat.

4. Compatibility conditions

We now require that the field equations (2.9)-(2.12) should be preserved under the action of the supergauge transformations (2.2)-(2.3). This is the procedure one follows in the massless case, and is a milder requirement with respect to the analysis of section 3.

If ν^B and $\lambda_{B'}$ are twistors, but not necessarily Killing spinors, they obey the equations (3.5) and (3.10), which imply that, for some independent spinor fields π^A and $\tilde{\pi}^{A'}$, one has

$$\nabla^{A'}_B \nu_C = \epsilon_{BC} \tilde{\pi}^{A'} \quad (4.1)$$

$$\nabla^A_{B'} \lambda_{C'} = \epsilon_{B'C'} \pi^A. \quad (4.2)$$

In the compatibility equations, whenever one has terms of the kind $S_{AA'} \nabla^A_{B'} \lambda_{C'}$, it is therefore more convenient to symmetrize and anti-symmetrize over B' and C' . A repeated use of this algorithm leads to a considerable simplification of the lengthy calculations. For example, the preservation condition of (2.9) has the general form

$$3f_2 \left(\nabla_{AA'} \nu^A + 2f_1 \lambda_{A'} \right) + \epsilon^{B'C'} \left[S_{AA'} \left(\nabla^A_{B'} \lambda_{C'} \right) + S_{AB'} \left(\nabla^A_{A'} \lambda_{C'} \right) \right] = 0. \quad (4.3)$$

By virtue of (4.2), equation (4.3) becomes

$$f_2 \left(\nabla_{AA'} \nu^A + 2f_1 \lambda_{A'} \right) + S_{AA'} \pi^A = 0. \quad (4.4)$$

Following (2.4)-(2.5), the action of the supercovariant derivative on $\pi_A, \tilde{\pi}_{A'}$ yields

$$S_{AA'} \pi_B \equiv \nabla_{AA'} \pi_B + f_1(\Lambda) \epsilon_{AB} \tilde{\pi}_{A'} \quad (4.5)$$

$$S_{AA'} \tilde{\pi}_{B'} \equiv \nabla_{AA'} \tilde{\pi}_{B'} + f_2(\Lambda) \epsilon_{A'B'} \pi_A. \quad (4.6)$$

Equations (4.4)-(4.5), jointly with the equations

$$\square \lambda_{A'} - 2\Lambda \lambda_{A'} = 0 \quad (4.7)$$

$$\nabla^{AA'} \pi_A = 2\Lambda \lambda^{A'} \quad (4.8)$$

which result from (4.2), lead to

$$(f_1 + f_2) \tilde{\pi}_{A'} + (f_1 f_2 - \Lambda) \lambda_{A'} = 0. \quad (4.9)$$

Moreover, the preservation of (2.10) under (2.2) leads to the equation

$$S^{B'(A} \pi^{B)} + f_2 \nabla^{B'(A} \nu^{B)} = 0 \quad (4.10)$$

which reduces to

$$\nabla^{B'(A} \pi^{B)} = 0 \quad (4.11)$$

by virtue of (4.1) and (4.5). Note that a supertwistor is also a twistor, since

$$S^{B'(A} \pi^{B)} = \nabla^{B'(A} \pi^{B)} \quad (4.12)$$

by virtue of the definition (4.5). It is now clear that, for a gauge freedom generated by twistors, the preservation of (2.11)-(2.12) under (2.3) leads to the compatibility equations

$$(f_1 + f_2)\pi_A + (f_1 f_2 - \Lambda)\nu_A = 0 \quad (4.13)$$

$$\nabla^{B(A'} \tilde{\pi}^{B')} = 0 \quad (4.14)$$

where we have also used the equation (see (3.6) and (4.1))

$$\nabla^{AA'} \tilde{\pi}_{A'} = 2\Lambda \nu^A. \quad (4.15)$$

Note that, if $f_1 + f_2 \neq 0$, one can solve (4.9) and (4.13) as

$$\tilde{\pi}_{A'} = \frac{(\Lambda - f_1 f_2)}{(f_1 + f_2)} \lambda_{A'} \quad (4.16)$$

$$\pi_A = \frac{(\Lambda - f_1 f_2)}{(f_1 + f_2)} \nu_A \quad (4.17)$$

and hence one deals again with Euclidean Killing spinors as in section 3. However, if

$$f_1 + f_2 = 0 \quad (4.18)$$

$$f_1 f_2 - \Lambda = 0 \quad (4.19)$$

the spinor fields $\tilde{\pi}_{A'}$ and $\lambda_{A'}$ become *unrelated*, as well as π_A and ν_A . This is a crucial point. Hence one may have $f_1 = \pm\sqrt{-\Lambda}$, $f_2 = \mp\sqrt{-\Lambda}$, and one finds a more general structure.

In the generic case, we do not assume that ν^B and $\lambda_{B'}$ obey any equation. This means that, on the second line of equation (4.3), it is more convenient to express the term in square brackets as $2S_{A(A'} \nabla^A_{B'}) \lambda_{C'}$. The rules of section 2 for the action of $S_{AA'}$ on spinors with many indices lead therefore to the compatibility conditions

$$3f_2 \left(\nabla_{AA'} \nu^A + 2f_1 \lambda_{A'} \right) - 6\Lambda \lambda_{A'} + 4f_1 \tilde{P}_{(A'B')}^{B'} + 3f_2 \tilde{Q}_{A'} = 0 \quad (4.20)$$

$$3f_1 \left(\nabla_{AA'} \lambda^{A'} + 2f_2 \nu_A \right) - 6\Lambda \nu_A + 4f_2 P_{(AB)}^B + 3f_1 Q_A = 0 \quad (4.21)$$

$$\Phi^{AB}_{C'D'} \lambda^{D'} + f_2 U^{(AB)}_{C'} - f_2 \nabla_{C'}^{(A} \nu^{B)} = 0 \quad (4.22)$$

$$\tilde{\Phi}^{A'B'}_{CD} \nu^D + f_1 \tilde{U}^{(A'B')}_C - f_1 \nabla_C^{(A'} \lambda^{B')} = 0 \quad (4.23)$$

where the detailed form of $P, \tilde{P}, Q, \tilde{Q}$ is not strictly necessary, but we can say that they do not depend explicitly on the trace-free part of the Ricci spinor, or on the Weyl spinors. Note that, in the massless limit $f_1 = f_2 = 0$, the equations (4.20)-(4.23) reduce to the familiar form of compatibility equations which admit non-trivial solutions only in Ricci-flat backgrounds [8].

Our consistency analysis still makes it necessary to set to zero $\Phi_{C'D'}^{AB}$ (and hence $\tilde{\Phi}_{CD}^{A'B'}$ by reality [11]). The remaining contributions to (4.20)-(4.23) should then become algebraic relations by virtue of the twistor equation. This is confirmed by the analysis of gauge freedom for secondary potentials in section 5.

5. Secondary potentials

In Ricci-flat 4-manifolds, secondary potentials for spin $\frac{3}{2}$ are introduced by requiring that locally [5,15]

$$\gamma_{A'B'}^C \equiv \nabla_{BB'} \rho_{A'}^{CB}. \quad (5.1)$$

The insertion of (5.1) into the Rarita-Schwinger equation (A.1) yields [5,8]

$$\epsilon_{FL} \nabla_{AA'} \nabla^{B'(F} \rho_{B'}^{A)L} + \frac{1}{2} \nabla_{A'}^A \nabla^{B'M} \rho_{B'(AM)} + \square_{AM} \rho_{A'}^{(AM)} + \frac{3}{8} \square \rho_{A'} = 0 \quad (5.2)$$

where $\rho_{A'} \equiv \rho_{A'C}^C$. Remarkably, equation (5.2) admits a square root in that, if the following equation holds [5,8,15]:

$$\nabla^{B'(F} \rho_{B'}^{A)L} = 0 \quad (5.3)$$

then (5.2) reduces to an identity by virtue of spinor Ricci identities jointly with the basic rules of two-spinor calculus [8]. However, if the trace-free part of the Ricci spinor vanishes but Λ does not vanish, the effect of Λ makes it necessary to write both (5.3) and the equation [5]

$$\rho_{A'} = 2\tilde{\alpha}_{A'} \quad (5.4)$$

where $\tilde{\alpha}_{A'}$ is a spinor field solving the Weyl equation [8,16]. An analogous local construction holds for the Γ -potentials. The corresponding secondary potentials are defined locally as

$$\Gamma_{AB}{}^{C'} \equiv \nabla_{BB'} \theta_A{}^{C'B'}. \quad (5.5)$$

The insertion of (5.5) into the Rarita-Schwinger equation (A.3) yields a second-order equation whose validity is ensured by the first-order equation [5]

$$\nabla^{B(F'} \theta_B{}^{A')L'} = 0 \quad (5.6)$$

jointly with [5]

$$\theta_A = 2\alpha_A \quad (5.7)$$

where $\theta_A \equiv \theta_{AC'}{}^{C'}$, and α_A solves the Weyl equation $\nabla^{AA'} \alpha_A = 0$ [8,16].

According to the prescription of section 2, which amounts to replacing $\nabla_{AA'}$ by $S_{AA'}$ in the field equations [9,10], we now *assume* that the super Rarita-Schwinger equations corresponding to (5.3) and (5.6) are (see section 7)

$$S^{B'(F} \rho_{B'}{}^{A)L} = 0 \quad (5.8)$$

$$S^{B(F'} \theta_B{}^{A')L'} = 0 \quad (5.9)$$

where the secondary potentials are subject locally to the supergauge transformations

$$\widehat{\rho}_{B'}{}^{AL} \equiv \rho_{B'}{}^{AL} + S_{B'}{}^A \mu^L \quad (5.10)$$

$$\widehat{\theta}_B{}^{A'L'} \equiv \theta_B{}^{A'L'} + S_B{}^{A'} \zeta^{L'}. \quad (5.11)$$

The analysis of the gauge freedom of the second kind is entirely analogous to the one in section 3, since equations like (2.4)-(2.5) now apply to μ_L and $\zeta_{L'}$. Hence we do not repeat this investigation.

A more general gauge freedom of the twistor type relies on the supertwistor equations (see (4.12))

$$S_{B'}^{(A} \mu^{L)} = \nabla_{B'}^{(A} \mu^{L)} = 0 \quad (5.12)$$

$$S_B^{(A'} \zeta^{L')} = \nabla_B^{(A'} \zeta^{L')} = 0. \quad (5.13)$$

Thus, on requiring the preservation of the super Rarita-Schwinger equations (5.8)-(5.9) under the supergauge transformations (5.10)-(5.11), one finds the preservation conditions

$$S^{B'(F} S_{B'}^{A)} \mu^L = 0 \quad (5.14)$$

$$S^{B(F'} S_B^{A')} \zeta^{L'} = 0 \quad (5.15)$$

which lead to

$$(f_1 + f_2)\pi_F + (f_1 f_2 - \Lambda)\mu_F = 0 \quad (5.16)$$

$$(f_1 + f_2)\tilde{\pi}_{F'} + (f_1 f_2 - \Lambda)\zeta_{F'} = 0. \quad (5.17)$$

Hence we can repeat the remarks following equations (4.16)-(4.19). Again, it is essential that π_F, μ_F and $\tilde{\pi}_{F'}, \zeta_{F'}$ may be unrelated if (4.18)-(4.19) hold. In the massless case, this is impossible, and hence there is no gauge freedom compatible with a non-vanishing cosmological constant [8].

If one does not assume the validity of equations (5.12)-(5.13), the general preservation equations (5.14)-(5.15) lead instead to the compatibility conditions

$$\psi^{AFL}_D \mu^D - 2\Lambda \mu^{(A} \epsilon^{F)L} + 2f_2 \omega^{(AF)L} + f_1 \epsilon^{L(A} T^{F)} + f_1 \epsilon^{L(A} S^{F)B'} \zeta_{B'} = 0 \quad (5.18)$$

$$\tilde{\psi}^{A'F'L'}_{D'} \zeta^{D'} - 2\Lambda \zeta^{(A'} \epsilon^{F')L'} + 2f_1 \tilde{\omega}^{(A'F')L'} + f_2 \epsilon^{L'(A'} \tilde{T}^{F')} + f_2 \epsilon^{L'(A'} S^{F')B} \mu_B = 0. \quad (5.19)$$

If we now combine the compatibility equations (4.20)-(4.23) with (5.18)-(5.19), and require that the gauge fields $\nu_A, \lambda_{A'}, \mu_A, \zeta_{A'}$ should not depend explicitly on the curvature of the background, we find that the trace-free part of the Ricci spinor has to vanish, and the Riemannian 4-geometry is forced to be conformally flat, since under our assumptions the equations

$$\psi_{AFLD} \mu^D = 0 \quad (5.20)$$

$$\tilde{\psi}_{A'F'L'D'} \zeta^{D'} = 0 \quad (5.21)$$

force the anti-self-dual and self-dual Weyl spinors to vanish. Remarkably, equations (5.20)-(5.21) are just the integrability conditions for the existence of non-trivial solutions of the supertwistor equations (5.12)-(5.13). Hence the spinor fields $\omega, T, \tilde{\omega}$ and \tilde{T} in (5.18)-(5.19) are such that these equations reduce to (5.16)-(5.17). In other words, for massive spin- $\frac{3}{2}$ potentials, the gauge freedom is indeed generated by solutions of the twistor equations in conformally flat Einstein 4-manifolds.

Last, on inserting the local equations (5.1) and (5.5) into the second half of the Rarita-Schwinger equations (cf (A.2) and (A.4) of the appendix), and then replacing $\nabla_{AA'}$ by $S_{AA'}$ [9], one finds equations whose preservation under the supergauge transformations (5.10)-(5.11) is again guaranteed if the supertwistor equations (5.12)-(5.13) hold.

6. Non-linear superconnection

As a first step in the proof that (5.8)-(5.9) arise naturally as integrability conditions of a suitable connection, we introduce a partial superconnection $W_{A'}$ (cf [15]) acting on unprimed spinor fields η_D defined on the Riemannian background.

With our notation [15]

$$W_{A'} \eta_D \equiv \eta^A S_{AA'} \eta_D - \eta_B \eta_C \rho_{A'}^{BC} \eta_D. \quad (6.1)$$

Writing

$$W_{A'} = \eta^A \Omega_{AA'} \quad (6.2)$$

where the operator $\Omega_{AA'}$ acts on spinor fields η_D , we obtain

$$\eta^A \Omega_{AA'} = \eta^A S_{AA'} - \eta_B \eta_C \rho_{A'}^{BC}. \quad (6.3)$$

Following [15], we require that $\Omega_{AA'}$ should provide a genuine superconnection on the spin-bundle, so that it acts in any direction. Thus, from (6.3) we can take (cf [15])

$$\Omega_{AA'} \equiv S_{AA'} - \eta^C \rho_{A'AC} = S_{AA'} - \eta^C \rho_{A'(AC)} + \frac{1}{2} \eta_A \rho_{A'}. \quad (6.4)$$

Note that (6.4) makes it necessary to know the trace $\rho_{A'}$, whilst in (6.1) only the symmetric part of $\rho_{A'}^{BC}$ survives. Thus we can see that, independently of the analysis in the previous sections, the definition of $\Omega_{AA'}$ picks out a potential of the Rarita-Schwinger type [15].

7. Integrability condition

In section 6 we have introduced a superconnection $\Omega_{AA'}$ which acts on a bundle with non-linear fibres, where the term $-\eta^C \rho_{A'AC}$ is responsible for the non-linear nature of $\Omega_{AA'}$ (see (6.4)). Following [15], we now pass to a description in terms of a vector bundle of rank three. On introducing the local coordinates (u_A, ξ) , where

$$u_A = \xi \eta_A \quad (7.1)$$

the action of the new operator $\tilde{\Omega}_{AA'}$ reads (cf [15])

$$\tilde{\Omega}_{AA'}(u_B, \xi) \equiv (S_{AA'} u_B, S_{AA'} \xi - u^C \rho_{A'AC}). \quad (7.2)$$

Now we are able to prove that (5.8)-(5.9) are integrability conditions.

The super β -surfaces are totally null two-surfaces whose tangent vector has the form $u^A \pi^{A'}$, where $\pi^{A'}$ is varying and u^A obeys the equation

$$u^A S_{AA'} u_B = 0 \quad (7.3)$$

which means that u^A is supercovariantly constant over the surface. On defining

$$\tau_{A'} \equiv u_B u_C \rho_{A'}^{BC} \quad (7.4)$$

the condition for $\tilde{\Omega}_{AA'}$ to be integrable on super β -surfaces is (cf [15])

$$u^A \tilde{\Omega}_{AA'} \tau^{A'} = u_A u_B u_C S^{A'(A} \rho_{A'}^{B)C} = 0 \quad (7.5)$$

by virtue of the Leibniz rule and of (7.2)-(7.4). Equation (7.5) implies

$$S^{A'(A} \rho_{A'}^{B)C} = 0 \quad (7.6)$$

which is the equation (5.8). Similarly, on studying super α -surfaces defined by the equation

$$\tilde{u}^{A'} S_{AA'} \tilde{u}_{B'} = 0 \quad (7.7)$$

one obtains (5.9). Thus, although (5.8)-(5.9) are naturally suggested by the local theory of spin- $\frac{3}{2}$ potentials, they have a deeper geometric origin, as shown.

8. Gauge invariance of boundary conditions

In the presence of boundaries one has to impose a suitable set of boundary conditions. We study the gauge invariance of locally supersymmetric boundary conditions first proposed in [18], which make it possible to relate bosonic and fermionic fields through the action of complementary projection operators at the boundary [8,19]. On using two-component spinor notation for supergravity [20-21], the spin- $\frac{3}{2}$ boundary conditions relevant for quantum cosmology and supergravity theories are [7,8,21]

$$\sqrt{2} \, e n_A^{A'} \psi_i^A = \pm \tilde{\psi}_i^{A'} \quad \text{at} \quad \partial M \quad (8.1)$$

where $e n_A^{A'}$ is the Euclidean normal to the boundary [5-8,21] and $(\psi_i^A, \tilde{\psi}_i^{A'})$ are the *independent* (i.e. not related by any conjugation) spatial components (hence $i = 1, 2, 3$) of

the spinor-valued one-forms appearing in the action functional of Euclidean supergravity [20,21]. In terms of the spatial components $e_{AB'i}$ of the tetrad, and of the primary potentials, $(\psi^A_i, \tilde{\psi}^{A'}_i)$ can be expressed as [3,8,20]

$$\psi_{A\ i} = \Gamma_{AB}^{C'} e_{C'i}^B \quad (8.2)$$

$$\tilde{\psi}_{A'\ i} = \gamma_{A'B'}^C e_C^{B'} e_{C'i}^B. \quad (8.3)$$

Bearing in mind that the gauge freedom is generated by solutions of the supertwistor equations (cf (4.1)-(4.2)), the boundary conditions (8.1) are preserved under the action of the supergauge transformations (2.2)-(2.3) if the spinor fields ν^C , $\lambda^{C'}$, π^C and $\tilde{\pi}^{C'}$ obey the boundary conditions

$$\sqrt{2} e n_A^{A'} \left(\tilde{\pi}^{C'} + f_1 \lambda^{C'} \right) e_{C'i}^A = \pm \left(\pi^C + f_2 \nu^C \right) e_C^{A'} e_{C'i}^A \quad \text{at} \quad \partial M. \quad (8.4)$$

Thus, we have obtained a simple algebraic relation among the spinor fields occurring in (4.1)-(4.2), which ensures the gauge invariance of the boundary conditions (8.1).

9. Concluding remarks

We have given an entirely two-spinor description of massive spin- $\frac{3}{2}$ potentials in Einstein 4-geometries. Although the supercovariant derivative (2.1) was well-known in the literature, following the work in [9], and its Lorentzian version was already applied in [13,17], the systematic analysis of primary and secondary potentials with the local form of their

supergauge transformations was not yet available in the literature, to the best of our knowledge.

Our first result is the two-spinor proof that, for massive spin- $\frac{3}{2}$ potentials, the gauge freedom is generated by solutions of the supertwistor equations in conformally flat Einstein 4-manifolds. Moreover, we have shown that the first-order equations (5.8)-(5.9), whose consideration is suggested by the local theory of massive spin- $\frac{3}{2}$ potentials, admit a deeper geometric interpretation as integrability conditions on super β - and super α -surfaces of a connection on a rank-three vector bundle. This result generalizes the analysis of massless spin- $\frac{3}{2}$ fields appearing in [15]. Besides that, in the presence of boundaries we have found the condition under which locally supersymmetric boundary conditions [18-19] are gauge-invariant. One now has to find explicit solutions of the equations (2.9)-(2.12), and the supercovariant form of β -surfaces studied in our paper deserves a more careful consideration. Hence we hope that our work can lead to a better understanding of twistor geometry and consistent supergravity theories in four-dimensions.

Appendix

For completeness, we write the Rarita-Schwinger equations for massless spin- $\frac{3}{2}$ potentials in Ricci-flat 4-manifolds. They take the form [1-5]

$$\epsilon^{B'C'} \nabla_{A(A'} \gamma^A_{B')C'} = 0 \quad (A.1)$$

$$\nabla^{B'(B} \gamma^A_{B'C')} = 0 \quad (A.2)$$

$$\epsilon^{BC} \nabla_{A'(A} \Gamma^{A'}_{B)C} = 0 \quad (A.3)$$

$$\nabla^{B(B'} \Gamma^{A')}_{BC} = 0. \quad (A.4)$$

Note that, if one works with $\nabla_{AA'}$ when Λ does not vanish, the right-hand sides of (A.1) and (A.3) should be replaced by $-3\Lambda \tilde{\alpha}_{A'}$ and $-3\Lambda \alpha_A$ respectively, where α_A and $\tilde{\alpha}_{A'}$ are spinor fields solving the Weyl equations [5,8,16]. In the massless case Λ is forced to vanish [1-4,8], but for massive models such contributions should be taken into account (cf our equations (2.9) and (2.11)).

In [13], the equation for Lorentzian Killing spinors is written in the form (see also equations (29) of [17], and cf our equations (3.1)-(3.2))

$$\nabla_{AX'} O_B = b \epsilon_{AB} \overline{O}_{X'} \quad (A.5)$$

where the parameter b is proportional to $\sqrt{-\Lambda}$, and the overbar denotes, as usual, the complex conjugation of spinors.

Acknowledgments

It is a pleasure to thank Paul Townsend for a useful conversation on massive spin- $\frac{3}{2}$ potentials. Our research was made possible in part by the European Union under the Human Capital and Mobility programme.

References

- [1] Penrose R 1991 Twistors as Spin- $\frac{3}{2}$ Charges *Gravitation and Modern Cosmology* eds A Zichichi, V de Sabbata and N Sánchez (New York: Plenum Press)
- [2] Penrose R 1991 *Twistor Newsletter* **32** 1
- [3] Penrose R 1991 *Twistor Newsletter* **33** 1
- [4] Penrose R and Mason L J *Twistor Newsletter* **37** 1
- [5] Esposito G 1995 *Complex General Relativity* (Fundamental Theories of Physics **69**) (Dordrecht: Kluwer)
- [6] Esposito G and Pollifrone G 1994 *Class. Quantum Grav.* **11** 897
- [7] Esposito G and Pollifrone G 1994 Twistors and Spin- $\frac{3}{2}$ Potentials in Quantum Gravity *Twistor Theory* ed S Huggett (New York: Marcel Dekker)
- [8] Esposito G, Gionti G, Kamenshchik A Yu, Mishakov I V and Pollifrone G *Spin- $\frac{3}{2}$ Potentials in Backgrounds with Boundary* (DSF preprint 95/20)
- [9] Townsend P K 1977 *Phys. Rev. D* **15** 2802
- [10] Izquierdo J M and Townsend P K 1995 *Class. Quantum Grav.* **12** 895
- [11] Penrose R and Rindler W 1986 *Spinors and Space-Time, Vol. 2: Spinor and Twistor Methods in Space-Time Geometry* (Cambridge: Cambridge University Press)
- [12] Penrose R and Rindler W 1984 *Spinors and Space-Time, Vol. 1: Two-Spinor Calculus and Relativistic Fields* (Cambridge: Cambridge University Press)

- [13] Siklos S T C 1985 Lobatchevski Plane Gravitational Waves *Galaxies, Axisymmetric Systems and Relativity* ed M A H MacCallum (Cambridge: Cambridge University Press)
- [14] Lewandowski J 1991 *Class. Quantum Grav.* **8** L11
- [15] Penrose R 1994 Twistors and the Einstein Equations *Twistor Theory* ed S Huggett (New York: Marcel Dekker)
- [16] Aichelburg P C and Urbantke H K 1981 *Gen. Rel. Grav.* **13** 817
- [17] Perry M J 1984 The Positive Mass Theorem and Black Holes *Asymptotic Behaviour of Mass and Spacetime Geometry* ed J Flaherty (Berlin: Springer)
- [18] Luckock H C and Moss I G 1989 *Class. Quantum Grav.* **6** 1993
- [19] Luckock H C 1991 *J. Math. Phys.* **32** 1755
- [20] D'Eath P D 1984 *Phys. Rev D* **29** 2199
- [21] Esposito G 1994 *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries* (Lecture Notes in Physics **m12**) (Berlin: Springer)